Review of Linear Algebra
Definitions

An $m \times n$ (read "m by n") **matrix**, is a rectangular array of entries, where $m$ is the number of rows and $n$ the number of columns.

$$A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}$$
Definitions (Con’t)

• \( A \) is \textit{square} if \( m = n \).
• \( A \) is \textit{diagonal} if all off-diagonal elements are 0, and not all diagonal elements are 0.
• \( A \) is the \textit{identity matrix} (\( I \)) if it is diagonal and all diagonal elements are 1.
• \( A \) is the \textit{zero} or \textit{null matrix} (\( 0 \)) if all its elements are 0.
• The \textit{trace} of \( A \) equals the sum of the elements along its main diagonal.
• Two matrices \( A \) and \( B \) are \textit{equal} iff they have the same number of rows and columns, and \( a_{ij} = b_{ij} \).
Some Basic Matrix Operations

• The *sum* of two matrices $A$ and $B$ (of equal dimension), denoted $A + B$, is the matrix with elements $a_{ij} + b_{ij}$.

• The *difference* of two matrices, $A - B$, has elements $a_{ij} - b_{ij}$.

• The *product*, $AB$, of $m \times n$ matrix $A$ and $p \times q$ matrix $B$, is an $m \times q$ matrix $C$ whose $(i,j)$-th element is formed by multiplying the entries across the $i$th row of $A$ times the entries down the $j$th column of $B$; that is,

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{pj}$$
Definitions (Con’t)

• The *transpose* $A^T$ of an $m \times n$ matrix $A$ is an $n \times m$ matrix obtained by interchanging the rows and columns of $A$.

• A square matrix for which $A^T = A$ is said to be *symmetric*.

• Any matrix $X$ for which $XA = I$ and $AX = I$ is called the *inverse* of $A$.

• Let $c$ be a real or complex number (called a *scalar*). The *scalar multiple* of $c$ and matrix $A$, denoted $cA$, is obtained by multiplying every elements of $A$ by $c$. If $c = -1$, the scalar multiple is called the *negative* of $A$. 
Determinant

• The *determinant* of a matrix $A$ is denoted $\det(A)$, $\det A$, or $|A|$.

• For a $2 \times 2$ matrix:

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

• For a $3 \times 3$ matrix:

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh.$$
Definitions (Con’t)

A **column vector** is an $m \times 1$ matrix:

$$
\mathbf{a} = \begin{bmatrix}
    a_1 \\
    a_2 \\
    \vdots \\
    a_m
\end{bmatrix}
$$

A **row vector** is a $1 \times n$ matrix:

$$
\mathbf{b} = [b_1, b_2, \ldots, b_n]
$$

A column vector can be expressed as a row vector by using the transpose:

$$
\mathbf{a}^T = [a_1, a_2, \ldots, a_m]
$$
Vector Norms

• The **norm** of a vector is

\[
\| \mathbf{x} \| = \left[ x_1^2 + x_2^2 + \cdots + x_m^2 \right]^{1/2}
\]

*(this is the 2-norm; other norms can be used)*

• This is recognized as the Euclidean distance from the origin to point \( \mathbf{x} \); or the length of a vector \( \mathbf{x} \).

• The norm can also be written as

\[
\| \mathbf{x} \| = \left[ \mathbf{x}^T \mathbf{x} \right]^{1/2}
\]
Some Basic Matrix Operations (Con’t)

The *inner product* (also called *dot product*) of two vectors

\[
a = \begin{bmatrix}
    a_1 \\
    a_2 \\
    \vdots \\
    a_m
\end{bmatrix} \quad b = \begin{bmatrix}
    b_1 \\
    b_2 \\
    \vdots \\
    b_m
\end{bmatrix}
\]

is defined as

\[
a^T b = b^T a = a_1 b_1 + a_2 b_2 + \cdots + a_m b_m = \sum_{i=1}^{m} a_i b_i.
\]

Note that the inner product is a scalar.
Vector Norms (Con’t)

Two vectors in $\mathbb{R}^m$ are \textit{orthogonal} if and only if their inner product is zero. Two vectors are \textit{orthonormal} if, in addition to being orthogonal, the length of each vector is 1.

An arbitrary vector $\mathbf{a}$ is turned into a vector $\mathbf{a}_n$ of unit length by performing the operation $\mathbf{a}_n = \mathbf{a}/|\mathbf{a}|$. Clearly, then, $|\mathbf{a}_n| = 1$.

A \textit{set of vectors} is said to be an \textit{orthogonal} set if every two vectors in the set are orthogonal.
Cross Product

• The **cross product** of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined only in three-dimensional space and is denoted by $\mathbf{a} \times \mathbf{b}$:

$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \sin(\theta) \mathbf{n}$$

• where $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{n}$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

• You can also compute the cross product via

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the standard basis vectors.
Combinations of Vectors

A **linear combination** of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an expression of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n$$

where the $\alpha$’s are scalars.

A vector $\mathbf{v}$ is **linearly dependent** on a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ if and only if $\mathbf{v}$ can be written as a linear combination of these vectors. Otherwise, $\mathbf{v}$ is **linearly independent** of the set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. 
Eigenvalues & Eigenvectors

**Definition:** The *eigenvalues* of a real matrix $\mathbf{M}$ are the real numbers $\lambda$ for which there is a nonzero vector $\mathbf{e}$ such that

$$\mathbf{Me} = \lambda \mathbf{e}.$$  

The *eigenvectors* of $\mathbf{M}$ are the nonzero vectors $\mathbf{e}$ for which there is a real number $\lambda$ such that $\mathbf{Me} = \lambda \mathbf{e}$. 

If $\mathbf{Me} = \lambda \mathbf{e}$ for $\mathbf{e} \neq 0$, then $\mathbf{e}$ is an *eigenvector* of $\mathbf{M}$ associated with *eigenvalue* $\lambda$, and vice versa. The eigenvectors are linearly independent.
Example: Consider the matrix

\[ M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \]

It is easy to verify that \( Me_1 = \lambda_1 e_1 \) and \( Me_2 = \lambda_2 e_2 \) for \( \lambda_1 = 1, \lambda_2 = 2 \) and

\[ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

In other words, \( e_1 \) is an eigenvector of \( M \) with associated eigenvalue \( \lambda_1 \), and similarly for \( e_2 \) and \( \lambda_2 \).